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New Exact Solutions for Time-Beta Fractional Boussinesq-Like Equations, Using the Sine-Cosine Method

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ABSTRACT:

The major goal of this paper is to extract exact travelling wave solutions and investigate the effects the fractional parameter on the dynamic response of soliton waves of four-time beta fractional Boussinesq-like equations. Sine-cosine method has been used to achieve explicit soliton solutions of these equations that emerged in coastal and ocean engineering. The obtained solutions have been studied in the form of hyperbolic and trigonometric functions. The behavior of some of the soliton solutions are demonstrated via 2D and 3D graphs. As a result of the fractional effects, physical changes are observed. The obtained results show that the proposed method is more convenient, powerful and efficient than other analytical approaches. The extracted results might improve our understanding of how waves propagate and could be beneficial to coastal and ocean engineering as well as other fields.

Keywords: Sine-cosine method; The Boussinesq- like equations; Time beta fractional derivative; the exact solutions.

1. INTRODUCTION

Mostly natural phenomena occurring in the universe are interpreted by nonlinear differential equations of integer order as well as non-integer order. Nonlinear fractional differential equations (NFDEs) respond promptly and effectively in a variety scientific and engineering fields, including dynamical systems, electromagnetic, technology, fusion plasma, viscoelastic, biology, signal processing, electrochemical, optical fiber, oceanography, solid state physics, geochemistry, finance, among others. Many intriguing aspects of fractional calculus and flexibility of fractional theory have captivated the attention of many researchers (Sadiya *et al.* 2022; Khantun *et al.*, 2022).

Fractional derivatives are defined in several ways, including conformable derivatives (Khalil *et al.*, 2014), Atangana-Baleanu derivatives (Atangana and Baleanu, 2016), beta derivatives (Yepez-Martinez *et al.*, 2018; Atangana *et al.*, 2016), M-truncated derivative (Sousa *et al.*, 2018), and Modified RL fractional derivatives (Jumarie, 2009). A fractional derivative can have a variety of properties, so that it is possible to use the one that is most suitable for the problem at hand. In fact, the study of fractional derivative operators is a popular topic. A lot of research has been done in this area, leading to an excessive number of findings e.g. see in (Ghanbari and Baleanu, 2020; Khater and Ghanbari, 2021; Ghanbari, 2019; Ghanbari *et al.*, 2019).

The areas that focus on the examination of wave patterns in the physical world are fascinating since they address an extremely scientific and developed idea of soliton waves. The soliton wave is an important nonlinear phenomenon. Solitons offer an intriguing perspective on nonlinear physical processes. Finding solitons of nonlinear phenomena has recently gained popularity among mathematicians and scientists. The primary advantage of soliton solutions is the fact that they can be used for evaluation equations with various kind of nonlinearities that are both integrable and non-integrable. The exact soliton solutions of fractional differential equations (FDEs) has been crucial in many kinds of physical science research. Consequently, several approaches have been devised for computing FDEs solutions. Among them are: the extended tanh-function method (Zaman *et al.*, 2022a, b), generalized $(\frac{G'}{G})$-expansion method (Uddin *et al.*, 2022), modified F- expansion method (Ahmad *et al.*, 2023), Sardar sub equation method (Ali *et al.*, 2023), improved F-expansion method (Akram *et al.*, 2023) extended simple equation method (Ahmad *et al.*, 2023), unified method (Ali *et al.*, 2023) and so on.

In this paper, we utilize the Sine-cosine technique introduced by Wazwaz (2004) to derive a set of new exact solutions for four distinct types of time beta fractional Boussinesq-like equations. For further context, relevant works include those by (Wazwaz, 2012), Elsami *et al.* (2014), Darvishi *et al.* (2017), Darvishi *et al.* (2018) and Osman (2019), each contributing to the understanding and development of this approach. The equations under consideration are expressed in the following forms:

$$D_{tt}^{2\beta}u - u_{xx} - (6u^2u_x + u_{xxx})_x = 0, \quad (1)$$

$$D_{tt}^{2\beta}u - u_{xx} - (6u^2u_x + D_{tt}^{2\beta}u_x)_x = 0, \quad (2)$$

$$D_{tt}^{2\beta}u - D_t^{\beta}u_x - (6u^2u_x + D_t^{\beta}u_{xx})_x = 0, \quad (3)$$

$$D_{tt}^{2\beta}u - (6u^2u_x + u_{xxx})_x = 0. \quad (4)$$

Where $D_t^{\beta}u$ is the time $\beta -$ fractional derivative of u (Atangana's comformable) order $0 < \beta < 1$ in time $t > 0, u_x = \frac{\partial u}{\partial x}$, $and\ u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Eq. (1) includes the fourth spatial derivative $u_{xxxx}$and the dissipative termu_{xx}. In Eq. (2), u_{xxxx} is replaced by a mixed spatial-temporal term.$D_{tt}^{2\beta}u_{xx}$. In Eq. (3), u_{xx} is replaced by$D_t^{\beta}u_x$ and u_{xxxx} is replaced by a spatial-temporal term$D_t^{\beta}u_{xx}$. Lastly, Eq. (4) lacks the dissipative term u_{xx}.

These equations (1)-(4) are non-integrable and are used as models in ocean and coastal sciences when $\beta = 1$. Some of these equations are employed in wave modeling and mathematical modeling of tidal oscillations.

This paper is structured as follows: A brief definition and properties of beta fractional derivative are given in section 2. Section 3 provides a detailed explanation of the Sine-cosine method. Section 4 discusses the application of the Sine-cosine method to the time β −fractional Boussinesq-like equations represented by equations (1)-(4) respectively. Section 5 offers a physical interpretation of some of the solutions obtained. Finally, Section 6 summarizes the main conclusions drawn from the study.

2. PROPERTIES OF BETA DERIVATIVE

Some properties of the Beta derivative are given using the following definition and theorem:

Definition (2.1), (Atangana *et al.*, 2016):

$$D_t^{\beta}(u(t)) = \lim_{\varepsilon \to 0} \frac{u\left(t + \varepsilon\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - u(t)}{\varepsilon},$$

Where $\Gamma(\beta) = \int_0^{\infty} t^{\beta-1}e^{-t}dt, \quad 0 < \beta < 1.$

and

$$D_t{}^{\beta}(u(t)) = \frac{d^{\beta}u(t)}{dt^{\beta}}.$$

Theorem: (2.1) Let $u(t)$ and $v(t)$ be $\beta -$ differentiable functions, for all $\beta \in (0,1)$ and $t > 0$. Then

(a) $D_t^{\beta}(au(t) + bv(t)) = aD_t^{\beta}u(t) + bD_t^{\beta}v(t), \forall\, a, b \in \mathbb{R}$,

(b)$D_t^{\beta}(u(t).v(t)) = u(t)D_t^{\beta}(v(t)) + v(t)D_t^{\beta}(u(t)),$

(c) $D_t^{\beta}\left(\frac{u(t)}{v(t)}\right) = \frac{v(t)D_t^{\beta}(u(t)) - u(t)D_t^{\beta}(v(t))}{v(t)^2},$

(d) $D_t^{\beta}(k) = 0$, where $k \in \mathbb{R}$,

(e) $D_t^{\beta}(u(t) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{du(t)}{dt}$.

Relevant works on this context of beta

fractional derivative see (Yusuf, *et al.*, 2019; Yiasir Arafat, *et al.*, 2023; Fiza *et al.*, 2024).

3. METHODOLOGY

In this section, we give a description of the sine-cosine method which is used to obtain exact solutions of partial differential equations.

3.1 The Sine-Cosine Method

Consider a nonlinear partial differential equation

$$P\left(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \cdots D_t^{\beta} u, D_x^{\beta} u, D_{tt}^{2\beta} u, D_{xx}^{2\beta} u, \ldots\right) = 0,\ 0 < \beta < 1. \quad (5)$$

Which describes the dynamical wave solution $u(x,t)$. The steps of the sine-cosine method has been proposed in (Wazwaz, 2004) as follows:

Step 1: To find the travelling wave solution of equation (5), we introduce

$$u(x,t) = U(z), \text{and} z = \varpi x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}. \quad (6)$$

Step 2: we use the following changes

$$\frac{\partial}{\partial t} = -\omega \frac{d}{dz}, \frac{\partial^2}{\partial t^2} = \omega^2 \frac{d^2}{dz^2}, \cdots, \frac{\partial}{\partial x} = \varpi \frac{d}{dz}, \frac{\partial^2}{\partial x^2} = \varpi^2 \frac{d^2}{dz^2} \ldots. \quad (7)$$

Now, utilizing Eq. (7) transforms the partial differential equation Eq. (5) into an ordinary differential equation:

$$Q(U, U', U'', \ldots) = 0, \quad (8)$$

Where U' denotes $\frac{du}{dz}$.

Step 3: Simplify Eq. (8) by integration if possible

Step 4: The solution will be expressed in the following form:

$$u(x,t) = \lambda sin^{\mu}(\vartheta z), \qquad |z| \le \frac{\pi}{\vartheta}, \quad (9)$$

or in the form

$$u(x,t) = \lambda cos^{\mu}(\vartheta z), |z| \le \frac{\pi}{2\vartheta}, \quad (10)$$

Where $\lambda, \vartheta, and\ \mu$ are parameters to be determined.

Step 5: Hence, the derivatives of Eq. (9) take the following form:

$$U(z) = \lambda sin^{\mu}(\vartheta z), \quad (11)$$

$$U^n(z) = \lambda^n sin^{n\mu}(\vartheta z), \quad (12)$$

$$U_z^n(z) = n\mu\vartheta\lambda^n \cos(\vartheta z)\, sin^{n\mu-1}(\vartheta z), \quad (13)$$

$$U_{zz}^n(z) = -n^2\vartheta^2\mu^2\lambda^2 sin^{n\mu}(\vartheta z) + n\vartheta^2\lambda^2\mu(n\mu - 1) sin^{n\mu-2}(\vartheta z). \quad (14)$$

And the derivative of Eq. (10) becomes

$$U(z) = \lambda cos^{\mu}(\vartheta z), \quad (15)$$

$$U^n(z) = \lambda^n cos^{n\mu}(\vartheta z), \quad (16)$$

$$U_{zz}^n(z) = -n^2\vartheta^2\mu^2\lambda^2 cos^{n\mu-1}(\vartheta z) + n\vartheta^2\lambda^2\mu(\mu - 1) cos^{n\mu-2}(\vartheta z). \quad (17)$$

And so on for further derivatives.

We substitute Eq. (10) to (17) into the previously obtained reduced equation Eq. (8), balancing the terms of the cosine functions when Eq. (10) is used, or balancing the sine functions when Eq. (9) is used. Finally, we solve the resulting system of algebraic equations with the assistance of computerized symbolic computation to determine all possible values of the parameters $\boldsymbol{\lambda}, \boldsymbol{\vartheta}, \boldsymbol{and}\ \boldsymbol{\mu}$.

4. APPLICATION

In this section, we utilize the sine-cosine method to derive exact solutions for four distinct β-time fractional Boussinesq-like equations.

4.1 The First $Time\beta$ − Fractional Boussinesq - like Equation

We consider The First Time β-Fractional Boussinesq-like Equation given by

$$D_{tt}^{2\beta} u - u_{xx} - (6u^2 u_x + u_{xxx})_x = 0. \quad (18)$$

By applying the following wave transformation:

$$u(x,t) = u(z),$$

Where $z = \varpi x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}$, (19)

Equation (18) reduces to ordinary differential equation:

$\omega^2 u'' - \varpi^2 u'' - \varpi(6u^2 u_x + u_{xxx})' = 0,$

$(\omega^2 - \varpi^2)u''(z) - \varpi(6\varpi u^2(z)u'(z) + \varpi^3 u'''(z))' = 0,$

(20)

Integrate once

$(\omega^2 - \varpi^2)u'(z) - \varpi(6\varpi u^2(z)u'(z) + \varpi^3 u'''(z)) = 0,$

(21)

Integrate again

$$(\omega^2 - \varpi^2)u(z) - \varpi\left(6\varpi.\frac{u^3}{3} + \varpi^3 u''(z)\right) = 0,$$

$(\omega^2 - \varpi^2)u(z) - 2\varpi^2 u^3(z) - \varpi^4 u''(z) = 0.$

(22)

Eq. (22) is the reduced ordinary differential equation.

Assume that Eq. (22) has a solution in the form of

$u(z) = \lambda\, sin^{\mu}(\vartheta\, z),$

(23)

$u'(z) = \mu\, \lambda\, \vartheta \cos(\vartheta\, z)\, .\, sin^{\mu-1}(\vartheta\, z),$

(24)

$u''(z) = \lambda\, \mu\, \vartheta\, (-\vartheta\, \mu sin^{\mu}(\vartheta\, z) + \vartheta\, (\mu - 1) sin^{\mu-2}(\vartheta\, z),$

$= -\lambda\, \mu^2 \vartheta^2\, sin^{\mu}(\vartheta\, z) + \lambda\, \mu\, \vartheta^2 (\mu - 1) sin^{\mu-2}(\vartheta\, z).$

(25)

For the cosine we have

$u(z) = \lambda\, cos^{\mu}(\vartheta\, z),$

(26)

$u'(z) = \lambda\, \mu\, cos^{\mu-1}(\vartheta\, z). -\vartheta\, sin^{\mu}(\vartheta\, z),$

(27)

$= -\lambda\, \mu\, \vartheta cos^{\mu-1}(\vartheta\, z) sin^{\mu}(\vartheta\, z),$

$u''(z) = -\lambda\, \mu^2 \vartheta^2 cos^{\mu}(\vartheta\, z) + \lambda\, \vartheta^2 \mu\, (\mu - 1) cos^{\mu-2}(\vartheta\, z).$

(28)

Substitute Eq. (23) and Eq. (25) in equation Eq. (22), we have

$(\omega^2 - \varpi^2)\, \lambda\, sin^{\mu}(\vartheta\, z) - 2\, \varpi^2 \lambda^3 sin^{3\mu}(\vartheta\, z) - \varpi^4\{-\lambda\, \mu^2 \vartheta^2 sin^{\mu}(\vartheta\, z)$
$+\lambda\, \mu\, \vartheta^2 (\mu - 1) sin^{\mu-2}(\vartheta\, z)\} = 0,$

(29)

$(\omega^2 - \varpi^2)\, \lambda\, sin^{\mu}(\vartheta\, z) - 2\, \varpi^2 \lambda^3 sin^{3\mu}(\vartheta\, z) + \varpi^4 \lambda\, \mu^2 \vartheta^2 sin^{\mu}(\vartheta\, z)$
$-\varpi^4 \lambda\, \mu\, \vartheta^2 (\mu - 1)\, sin^{\mu-2}(\vartheta\, z) = 0.$

(30)

Eq. (30) is fulfilled if the following system of algebraic equations is satisfied:

$\mu - 1 \neq 0,$

$3\, \mu = \mu - 2,$

$-2\, \varpi^2 \lambda^3 - \varpi^4 \lambda\, \mu\, \vartheta^2 (\mu - 1) = 0,$

$(\omega^2 - \varpi^2)\, \lambda + \varpi^4 \lambda\, \mu^2\, \vartheta^2 = 0.$

Solving the system using Maple, we obtained

$\mu = -1,\ \lambda = \pm\frac{\sqrt{\varpi^2-\omega^2}}{\varpi},\ \vartheta = \pm\frac{\sqrt{\varpi^2-\omega^2}}{\varpi^2}.$

(31)

Using the cosine method given by Eq. (26) will yield the same results.

Given Eq. (31), the solutions obtained from Eq. (23) and Eq. (19) are as follows:

$u_{11}(x,t) = \frac{\sqrt{\varpi^2-\omega^2}}{\varpi} \csc\left[\frac{\sqrt{\varpi^2-\omega^2}}{\varpi^2}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right].$

(32)

And for cosine function, we have

$u_{12}(x,t) = \frac{\sqrt{\varpi^2-\omega^2}}{\varpi} \sec\left[\frac{\sqrt{\varpi^2-\omega^2}}{\varpi^2}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right].$

(33)

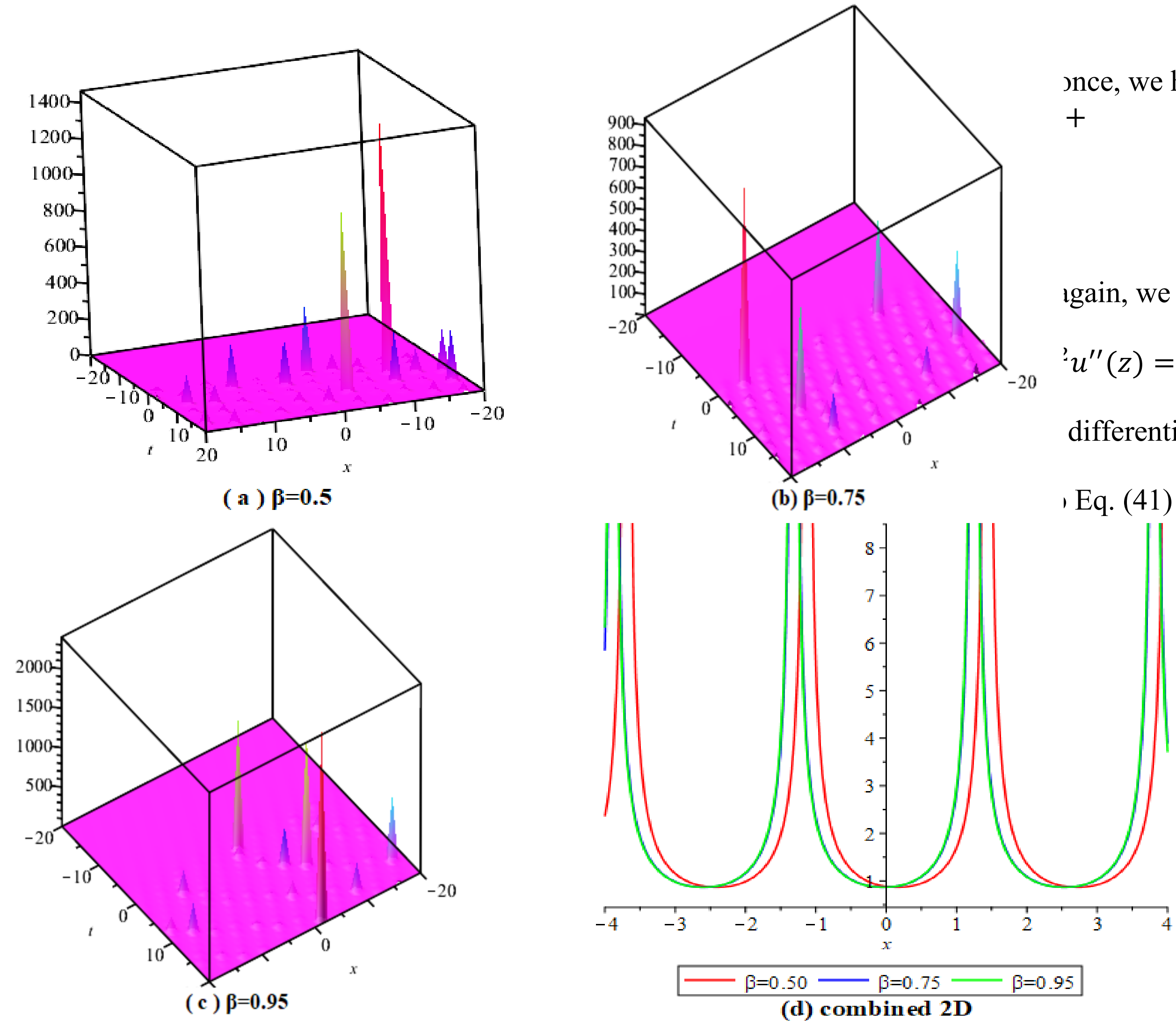
Eq. (32) and Eq. (33) are valid only if $\varpi^2 - \omega^2 > 0$, *so that* $\omega^2 < \varpi^2,\quad i = \sqrt{-1}.$

However, for $\omega^2 > \varpi^2$ we obtain the following solutions

$u_{13}(x,t) = -\frac{\sqrt{\varpi^2-\omega^2}}{\varpi} \operatorname{csch}\left[\frac{\sqrt{\omega^2-\varpi^2}}{\varpi^2}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right],$ (34)

$u_{14}(x,t) = \frac{\sqrt{\varpi^2-\omega^2}}{\varpi} \operatorname{sech}\left[\frac{\sqrt{\omega^2-\varpi^2}}{\varpi^2}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right].$

(35)



…once, we have
+

…again, we
$u''(z) = 0.$
…differential
…Eq. (41) gives

FIGURE 1. The graphical illustration for beta fractional derivative of solution $|u_{11}(x,t)|$ with $\varpi = 2, \omega = 1, t = 1.5, \mu = 1$.**(a)** $\beta = 0.5$, $(\boldsymbol{b})\beta = 0.75$, $(\boldsymbol{c})\beta = 0.95$, $(\boldsymbol{d})$Combined 2Dgraph for distinct values of β.

4.2 The Second Time β-Fractional Boussinesq-like Equation

In this section, we investigate the second β-time fractional Boussinesq-like equation:

$$D_{tt}^{2\beta} u - u_{xx} - (6u^2 u_x + D_{tt}^{2\beta} u_x)_x = 0. \quad (36)$$

We use the wave transformation

$$u(x,t) = u(z),$$

$$z = \varpi x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}. \quad (37)$$

Eq. (35) reduces to

$$\omega^2 u'' - \varpi^2 u'' - \varpi(6u^2 u_x + u_{xtt})' = 0, \quad (38)$$

$$(\omega^2 - \varpi^2)u''(z) - \varpi(6\varpi u^2(z)u'(z) +$$

(44)

Solving system Eq. (44), we obtain

$$\mu = -1, \qquad \lambda = \pm\frac{\sqrt{\omega^2 - \varpi^2}}{\varpi},$$

$$\vartheta = \pm\frac{\sqrt{\varpi^2 - \omega^2}}{\varpi\omega},$$

Similar results are also achieved when employing the cosine function.

As a result, the following solutions were derived:

(48)

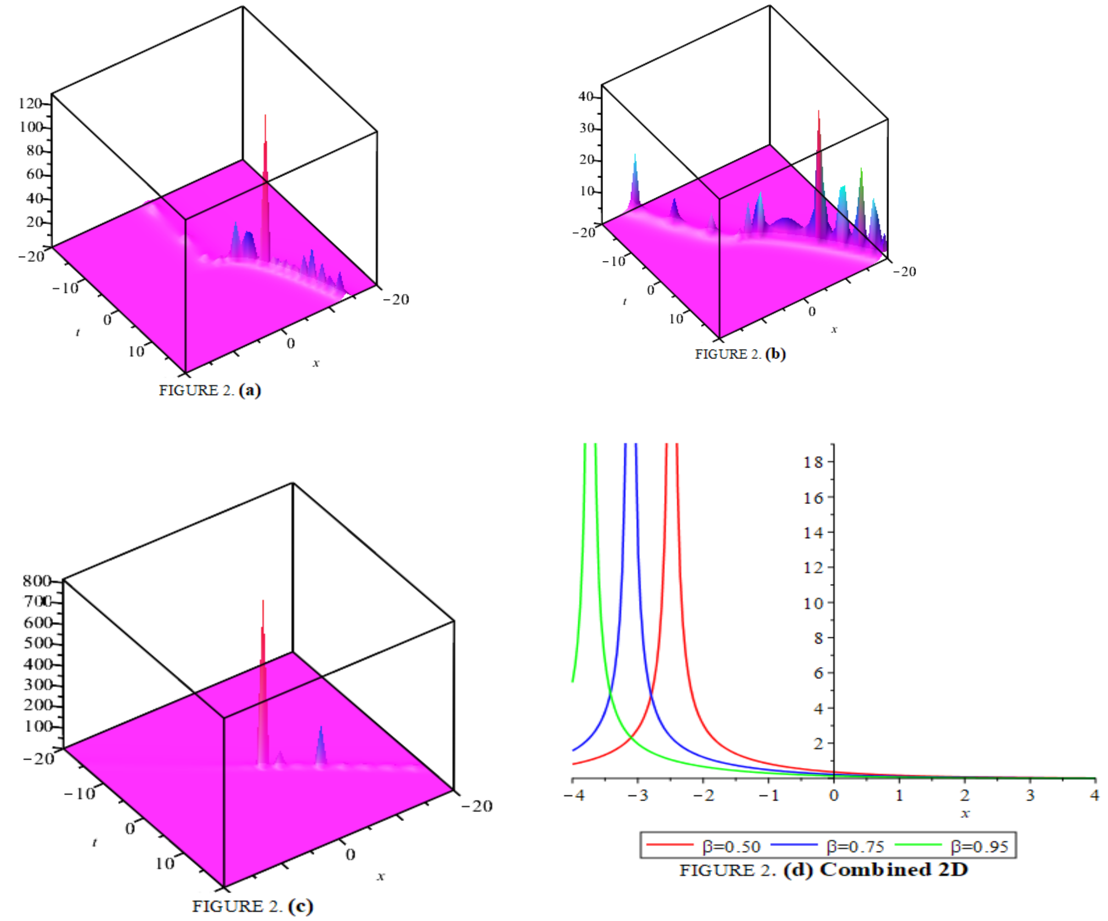


FIGURE 2. The graphical illustration for beta fractional derivative of solution $|u_{21}(x,t)|$ with $\varpi = 2, \omega = -3, t = 1.5,\ \mu = -1.$ **(a)** $\beta = 0.5$, **(b)** $\beta = 0.75$, **(c)** $\beta = 0.95$, **(d)** Combined 2Dgraph for distinct values of β.

$$u_{21}(z) = \frac{\sqrt{\omega^2-\varpi^2}}{\varpi}\csc\left[\frac{\sqrt{\varpi^2-\omega^2}}{\varpi\,\omega}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right], \omega^2 < \varpi^2, \quad (45)$$

$$u_{22}(z) = \frac{\sqrt{\omega^2-\varpi^2}}{\varpi}\sec\left[\frac{\sqrt{\varpi^2-\omega^2}}{\varpi\,\omega}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right], \quad \omega^2 < \varpi^2. \quad (46)$$

For $\omega^2 > \varpi^2$ we also obtained the following solutions:

$$u_{23}(z) = -i\frac{\sqrt{\omega^2-\varpi^2}}{\varpi}\operatorname{csch}\left[\frac{\sqrt{\omega^2-\varpi^2}}{\varpi\,\omega}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right], \quad (47)$$

$$u_{24}(z) = \frac{\sqrt{\omega^2-\varpi^2}}{\varpi}\operatorname{sech}\left[\frac{\sqrt{\omega^2-\varpi^2}}{\varpi\,\omega}\left(\varpi\, x + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right].$$

4.3 The Third Time β −Fractional Boussinesq - like Equation

In this section we study the third time beta-fractional Boussinesq- like equation:

$$D_{tt}^{2\beta}u - u_{xt}^{\beta} - (6u^2u_x + D_t^{\beta}u_{xx})_x = 0. \quad (49)$$

Using the travelling wave transformation:

$$u(x,t) = u(z),$$

$$z = \varpi x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}.$$

Substituting these in Eq. (49) gives

$$(\omega^2 + \varpi\omega)u''(z) - \varpi\left(6u^2(z)u'(z) - \varpi^2\omega u'''(z)\right)' = 0,$$

(50)
Integrating twice and setting the constants of integration to zero, we obtain:
$(\omega^2+\varpi\omega)u(z)-2\varpi^2u^3(z)+\varpi^3\omega u''(z)=0.$
(51)
Assuming that Eq. (51) has a solution in the form:
$$u(z)=\lambda\sin^{\mu}(\vartheta\, z),$$
Where ω and ϖ are constants, substituting Eq. (23) and Eq. (25) into Eq. (51) yields:
$(\omega^2+\varpi\omega)\lambda\sin^{\mu}(\vartheta\, z)-2\varpi^2(\lambda\sin^{\mu}(\vartheta\, z))^3+\varpi^3\omega\left(-\lambda\,\mu^2\vartheta^2\,sin^{\mu}(\vartheta\, z)+\lambda\,\mu\,\vartheta^2(\mu-1)sin^{\mu-2}(\vartheta\, z)\right)=0,$
(52)
$(\omega^2+\varpi\omega)\lambda\sin^{\mu}(\vartheta\, z)-2\varpi^2\lambda^3\sin^{3\mu}(\vartheta\, z)-\varpi^3\omega\lambda\mu^2\vartheta^2\sin^{\mu}(\vartheta\, z)+\varpi^3\omega\lambda\mu\vartheta^2(\mu-1)\sin^{\mu-2}(\vartheta\, z)=0.$
(53)
By equating the exponents and coefficients of each pair of sine functions to zero, we derive the following system of algebraic equations:
$$\mu-1\neq 0,$$
$$3\mu=\mu-2,$$
$(\omega^2+\varpi\omega)\lambda-\varpi^3\omega\lambda\mu^2\vartheta^2=0,$
(54)
$$-2\varpi^2\lambda^3+\varpi^3\omega\lambda\mu\vartheta^2(\mu-1)=0.$$
Solving the system Eq. (54) yields
$\mu=-1,\ \lambda=\pm\frac{\sqrt{\omega^2+\omega\varpi}}{\varpi},\ \vartheta=\pm\frac{\sqrt{\varpi(\omega+\varpi)}}{\varpi^2}.$
(55)
Consequently, for $\varpi\omega+\varpi^2>0$, we obtained the following solutions
$u_{31}(z)=\frac{\sqrt{\omega^2+\omega\varpi}}{\varpi}\csc\left[\frac{\sqrt{\varpi(\omega+\varpi)}}{\varpi^2}\left(\varpi\, x-\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right],$
(56)
and
$u_{32}(z)=\frac{\sqrt{\omega^2+\omega\varpi}}{\varpi}\sec\left[\frac{\sqrt{\varpi(\omega+\varpi)}}{\varpi^2}\left(\varpi\, x-\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right].$
(57)
However, for $\varpi\omega+\varpi^2<0$, gives the following solutions
$u_{33}(z)=-i\frac{\sqrt{-\omega^2-\omega\varpi}}{\varpi}\operatorname{csch}\left[\frac{\sqrt{-\varpi(\omega+\varpi)}}{\varpi^2}\left(\varpi\, x-\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right],$ (58)
and
$u_{34}(z)=\frac{\sqrt{-\omega^2-\omega\varpi}}{\varpi}\operatorname{sech}\left[\frac{\sqrt{-\varpi(\omega+\varpi)}}{\varpi^2}\left(\varpi\, x-\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right].$ (59)

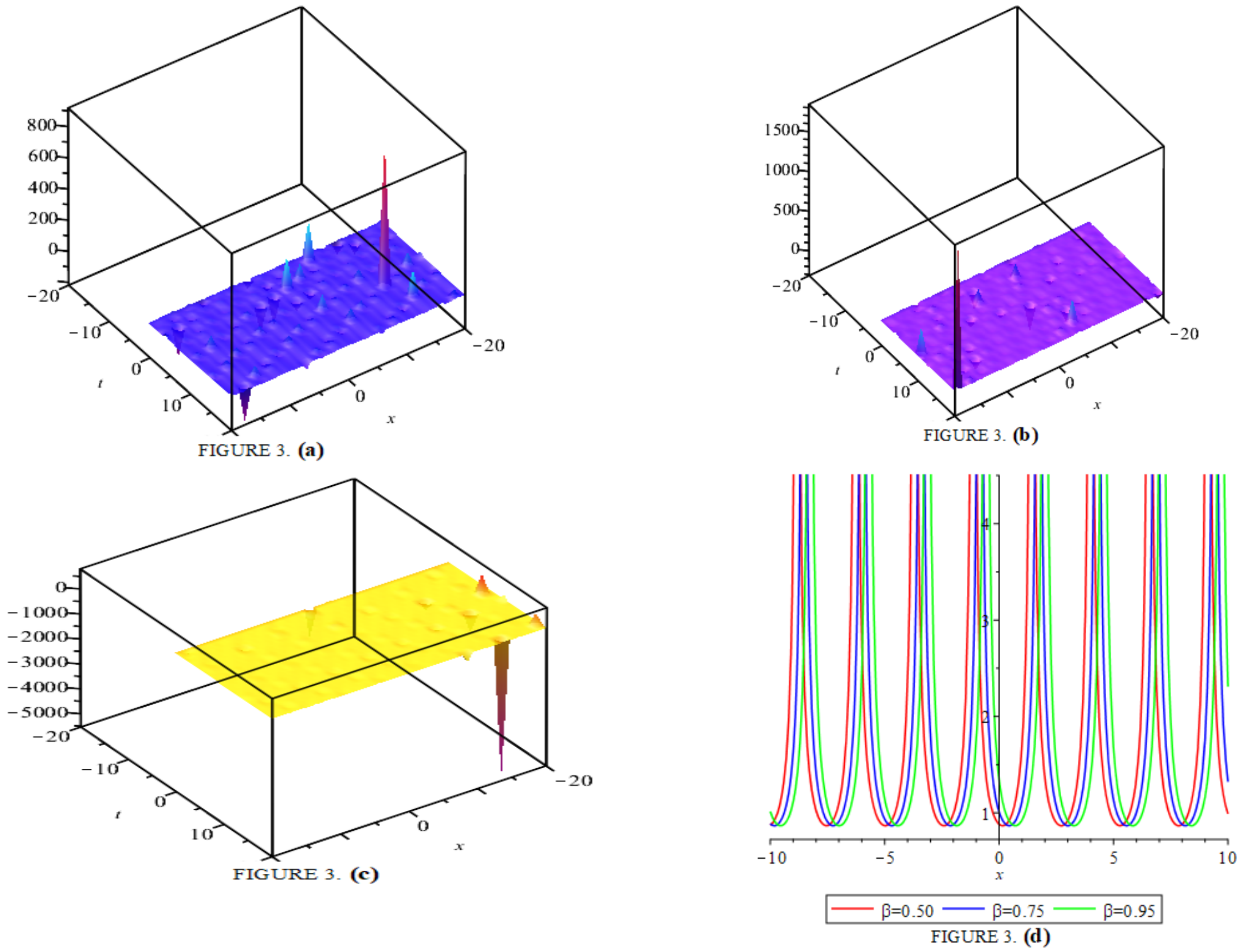


FIGURE 3. The graphical illustration for beta fractional derivative of solution $|u_{31}(x,t)|$ with $\varpi = 2, \omega = 1, t = 1.5$, $\mu = -1$.**(a)** $\beta = 0.5$, $(\boldsymbol{b})\beta = 0.75$, $(\boldsymbol{c})\beta = 0.95$, $(\boldsymbol{d})$Combined 2D graph for distinct values of β.

4.4 The Fourth Time $\boldsymbol{\beta}$ −Fractional Boussinesq- like Equation

In this section we study the fourth time beta-fractional Boussinesq like equation:

$$D_{tt}^{2\beta} u - (6u^2 u_x + u_{xxx}^{\beta})_x = 0.$$
(60)

Using the travelling wave transformation:

$$u(x,t) = u(z),$$
$$z = \varpi x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}.$$

Substituting these in Eq. (60) becomes

$$\omega^2 u''(z) - \varpi(6u^2 u_x + u_{xxx})' = 0,$$
(61)

$$\omega^2 u''(z) - \varpi(6u^2 \varpi u'(z) + \varpi^3 u'''(z)' = 0,$$
(62)

Integrating Eq. (62) once, we have

$$\omega^2 u'(z) - \varpi(6\varpi u^2(z) u'(z) + \varpi^3 u'''(z) = 0,$$
(63)

$$\omega^2 u'(z) - 6\varpi^2 u^2(z) u'(z) - \varpi^4 u'''(z) = 0,$$
(64)

Integrating Eq. (64) again gives

$$\omega^2 u(z) - 2\varpi^2 u^3(z) - \varpi^4 u''(z) = 0.$$
(65)

Where ϖ *and* ω are constants. Substituting Eq. (23) and Eq. (25) into Eq. (65) gives

$$\omega^2 \lambda \sin^{\mu}(\vartheta\, z) - 2\varpi^2 \lambda^3 \sin^{3\mu}(\vartheta\, z) - \varpi^4\left(-\lambda\, \mu^2 \vartheta^2 \sin^{\mu}(\vartheta\, z) + \lambda\, \mu\, \vartheta^2 (\mu - 1) sin^{\mu-2}(\vartheta\, z)\right) = 0,$$
(66)

$$\omega^2 \lambda \sin^{\mu}(\vartheta\, z) - 2\varpi^2 \lambda^3 \sin^{3\mu}(\vartheta\, z) + \varpi^4 \lambda\, \mu^2 \vartheta^2 \sin^{\mu} \vartheta\, z) - \varpi^4 \lambda \mu \vartheta^2 (\mu - 1) \sin^{\mu-2}(\vartheta\, z) = 0.$$
(67)

By setting the exponents and coefficients of each pair of sine functions to zero, we derive the following system of algebraic equations:

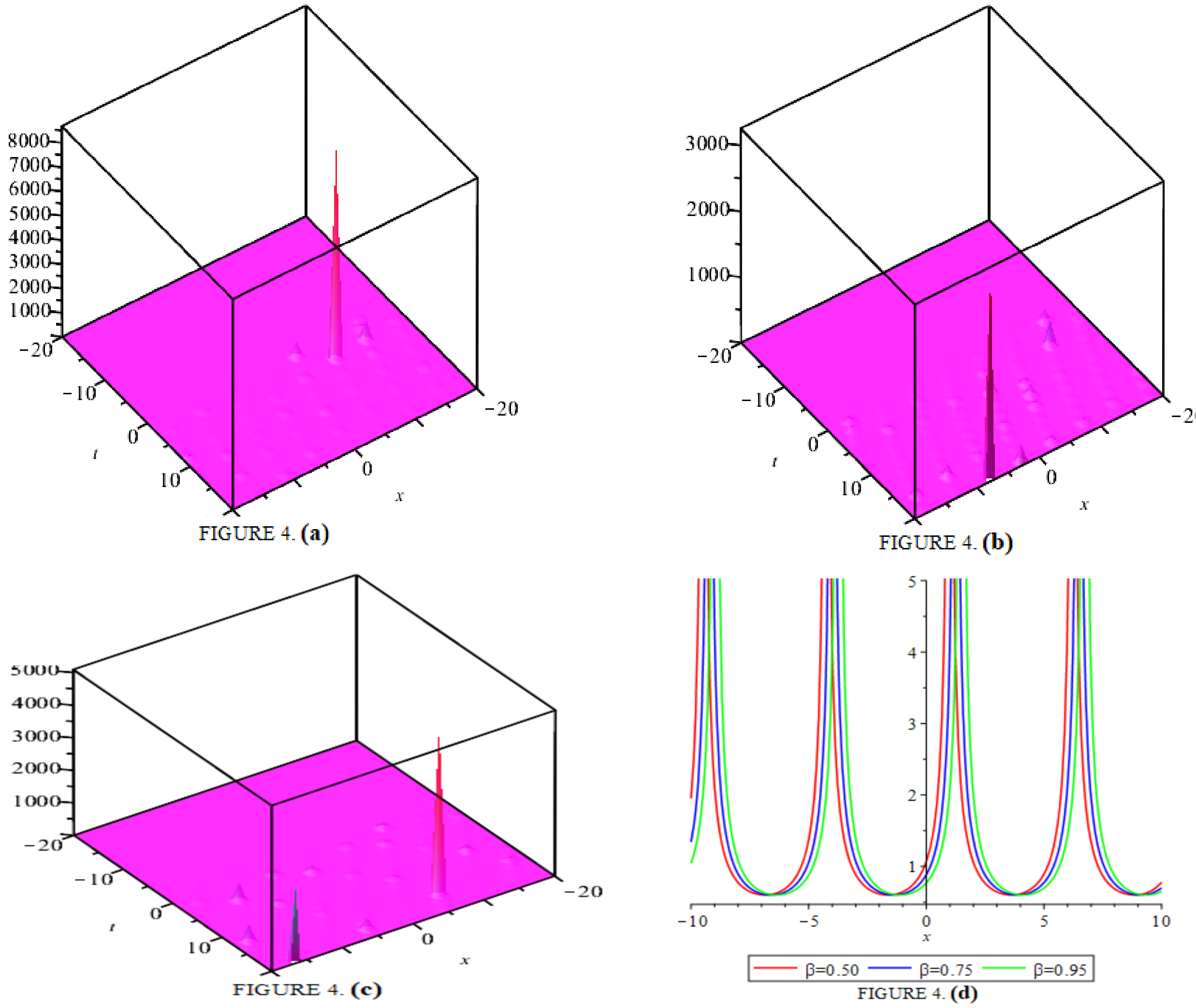


FIGURE 4. The graphical illustration for beta fractional derivative of solution $|u_{41}(x,t)|$ with $\varpi = 5, \omega = 3, t = 1.5,\ \mu = -1$.**(a)** $\beta = 0.5$, $(\boldsymbol{b})\beta = 0.75$, $(\boldsymbol{c})\beta = 0.95$, $(\boldsymbol{d})$Combined 2D graph for distinct values of β.

$$\mu - 1 \neq 0,$$
$$3\mu = \mu - 2,$$
$$\omega^2\lambda + \varpi^4\lambda\,\mu^2\vartheta^2 = 0,$$
(68)
$$-2\varpi^2\lambda^3 - \varpi^4\lambda\mu\vartheta^2(\mu - 1) = 0.$$

Upon solving the system of equations Eq. (68), we obtain:

$$\mu = -1, \quad \lambda = \pm\frac{\omega}{\varpi}, \quad \vartheta = \pm\frac{\omega i}{\varpi^2}.$$
(69)

Consequently, we obtained the following solutions

$$u_{41}(z) = \frac{\omega}{\varpi}\csc\left[\frac{\omega i}{\varpi^2}\left(\varpi x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right],$$
(70)

and

$$u_{42} = \frac{\omega}{\varpi}\sec\left[\frac{\omega i}{\varpi^2}\left(\varpi x - \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)\right)\right].$$
(71)

5 PHYSICAL EXPLANATION

In this section, we generated various types of exact solutions for Equations (1)-(4) using the Sine-cosine approach. These solutions are in the form of trigonometric and hyperbolic functions.

Fig 1. (a)-(c) Depicts modulus of solutions for $|u_{11}(x,t)|$, Eq. (32) with various parameter values, $\varpi = 2,\ \omega = 1,\ \mu = -1,\ \lambda = \frac{\sqrt{3}}{2},\ \vartheta = \frac{\sqrt{6}}{4}$, as well as the beta fractional order values, $\beta = 0.5\ for\ (a)$, $\beta = 0.75\ for\ (b)\ and\ \beta = 0.95\ for\ (c)$. We observed multiple solitons with varying heights and positions. (d) Shows the effect of beta fractional order values in combined 2D wave profile with $\beta = 0.5,\ 0.75,\ 0.95\ and\ t = 1.5$. We observed the wave propagates in the x-direction for increasing values of

β. For specific values of the parametres $\varpi = 2$, $\omega = -3$, $\mu = -1$, $\lambda = \frac{\sqrt{5}}{2}$, $\vartheta = -\frac{\sqrt{5}}{6}I$, the 3D wave profile of the solution $|u_{2,1}(x,t)|$ which depicts periodic solitary wave corresponding to Fig 2.(a)-(c) with different beta values $\beta = 0.5\ for\ (a),\ \beta = 0.75\ for\ (b)\ and\ \beta = 0.95\ for\ (c)$.. Moreover, the combined 2D graph is shown in Fig. 2(d) with distinct beta values, $\beta = 0.5,\ 0.75,\ 0.95\ and\ t = 1.5$.

Fig. 3(a)-(c) represents the periodic wave structure of the solution $|u_{3,1}(x,t)|$ for the parameter values $= 2$, $\omega = 1$, $\mu = -1, \lambda = \frac{\sqrt{3}}{2}$, $\vartheta = \frac{\sqrt{6}}{4}$, with $\beta = 0.5\ for\ (a),\ \beta = 0.75\ for\ (b)\ and\ \beta = 0.95\ for\ (c)$and the 2D graph is plotted in Fig. 3(d) with beta values $\beta = 0.5,\ 0.75,\ 0.95\ and\ t = 1.5$, which shows the effect of beta fractional order. Fig. 4(a)-(c) illustrates the 3D wave structure of the solution $|u_{4,1}(x,t)|$ representing singular soliton solution with parameter values $\varpi = 5$, $\omega = 3$, $\mu = -1$, $\lambda = \frac{3}{5}$, $\vartheta = 3/25$, and for specific values of the beta: $\beta = 0.5\ for\ (a),\ \beta = 0.75\ for\ (b)\ and\ \beta = 0.95\ for\ (c)$. The 2D combined wave profiles are shown in Fig. 4(d) for $\beta = 0.5,\ 0.75,\ 0.95\ and\ t = 1.5$ to show the effect of beta fractional order. From the above discussion, it is obvious that the sine-cosine approach can provide different types of wave structures to fractional Boussinesq-like equations with beta derivative for various values of parameters.

The obtained solutions derived through this method play a crucial role in understanding the structure and dynamic behavior of the problem at hand. The method proposed in this work is straightforward, dependable, and effective, with the potential to be extended for studying and solving numerous other nonlinear evolution equations across various scientific and engineering disciplines.

CONCLUSION

In this paper, we employed the sine-cosine method to obtain travelling wave solutions for four distinct Beta fractional Boussinesq-like equations. The obtained results include singular traveling wave soliton solution, periodic soliton solutions and multiple soliton solution which involve trigonometric functions, hyperbolic functions and complex solutions as well. We examined all constraints that guarantee the existence of these new exact solutions. The results obtained from this study are anticipated to be highly beneficial in numerous areas of mathematical physics, costal engineering and applied mathematics, including fluid dynamics, nonlinear optics, plasma physics, and other related fields.

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